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Quaternion Linear Canonical Transform Application

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Abstract

The quaternion linear canonical transform (QLCT) is a generalization of the classical linear canonical transform (LCT) using quaternion algebra. The focus of this paper is to introduce an application of the QLCT to study of generalized swept-frequency filters.

Keywords: quaternion linear canonical transform, swept-frequency filters

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1. Introduction

The linear canonical transform (LCT) has been studied extensively in the literature. It plays an important role in many field of optics and signal processing [1, 5, 6]. The LCT can be regarded as generalization of many mathematical transforms such as the Fourier transform, Laplace transform, the fractional Fourier transform, the Fresnel transform and the other transforms. Many useful properties of this extended transform are already known, including shift, modulation, convolution, correlation and uncertainty principle (see, e.g., [7] and the references therein).

Recently in [4], the authors introduced the quaternion linear canonical transform (QLCT), which is generalization of the LCT in the framework of quaternion algebra. Several fundamental properties of the generalized transform such as the Parseval's formula and uncertainty principle are established. Here we provide an application of the QLCT to study of generalized swept-frequency filters.

2. Quaternion

The quaternion, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843. It is a generalization of complex number to a 4D algebra and is denoted by $\mathbb{H}$. Every element of H can be written in a hypercomplex form as follows

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3; q_0, q_1, q_2, q_3 \in \mathbb{R}\}. \quad (1)$$

Here the three different imaginary pats obey the following multiplication rules:

$$\boldsymbol{ij} = -\boldsymbol{ji} = -\boldsymbol{k}, \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \boldsymbol{i}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1, \tag{2}$$

For a quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in \mathbb{H}$, q_0 is called the scalar part of q denoted by $Sc(q)$ and a pure quaternion $\boldsymbol{q}$ denoted by Vec $(q) = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$.

Any quaternion q may be split up into

$$q = q_+ + q_-, q_\pm = \frac{1}{2}(q \pm \boldsymbol{i}q\boldsymbol{j}). \tag{3}$$

Above gives

$$q_\pm = \{q_0 \pm q_3 + \boldsymbol{i}(q_1 \mp q_2)\}\frac{1 \pm \boldsymbol{k}}{2}. \tag{4}$$

Or, equivalently,

$$q_\mp = \{q_0 \mp q_3 + \boldsymbol{i}(q_1 \pm q_2)\}\frac{1 \mp \boldsymbol{k}}{2}. \tag{5}$$

The proof of equation (4) easily can be seen from

$$\begin{aligned}
q_\pm &= \frac{1}{2}(q \pm \boldsymbol{i}q\boldsymbol{j}) \\
&= \frac{1}{2}\{(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \pm \boldsymbol{i}(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3)\boldsymbol{j}\} \\
&= \frac{1}{2}\{(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \pm (\boldsymbol{k}q_0 + \boldsymbol{ik}q_1 + \boldsymbol{kj}q_2 + q_3)\} \\
&= \frac{1}{2}\{(q_0 \pm \boldsymbol{k}q_0) + (\boldsymbol{i}q_1 \pm \boldsymbol{ik}q_1) + (\boldsymbol{j}q_2 \pm \boldsymbol{kj}q_2) + (\boldsymbol{k}q_3 \pm q_3)\} \\
&= \frac{1}{2}\{(q_0 \pm \boldsymbol{k}q_0) + (\boldsymbol{i}q_1 \pm \boldsymbol{ik}q_1) + ((-\boldsymbol{ik})q_2 \pm (-\boldsymbol{i})q_2) + (\boldsymbol{k}q_3 \pm q_3)\} \\
&= \frac{1}{2}\{q_0(1 \pm \boldsymbol{k}) + \boldsymbol{i}q_1(1 \pm \boldsymbol{k}) \mp \boldsymbol{i}q_2(1 \pm \boldsymbol{k}) \pm q_3(1 \pm \boldsymbol{k})\} \\
&= ((q_0 \pm q_3) + \boldsymbol{i}(q_1 \mp q_2))\frac{(1 \pm \boldsymbol{k})}{2}.
\end{aligned}$$

3. Quaternion Linear Canonical Transform

Since the quaternion multiplication is not commutative, the definition of the quaternion linear canonical transform (QLCT) is not unique. In this part, we firstly begin with the definition of the QLCT.

Definition 3.1 (QLCT). $Let\, A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ be two parameters statisfying det $(A_1) = a_s d_s - b_s c_s = 1, s = 1,2$. the QLCT of a quaternion signal $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is defined by

$$L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{\mathbb{R}^2} f(x) K_{A_1}(x_1,\omega_1) K_{A_2}(x_2,\omega_2) d\boldsymbol{x}, \; b_1 b_2 \neq 0 \\ \sqrt{d_1 d_2} f(d_1\omega_1, d_2\omega_2) e^{\mu\left(\frac{c_1 d_1}{2}\right){\omega_1}^2} e^{\mu\left(\frac{c_2 d_2}{2}\right){\omega_2}^2}, b_1 b_2 = 0. \end{cases} \tag{6}$$

The kernel of the transform is given by, respectively,

$$K_{A_1}(x_1,\omega_1) = \frac{1}{\sqrt{2\pi\mu b_1}} e^{\frac{1}{2}\mu\left(\frac{a_1}{b_1}{x_1}^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}{\omega_1}^2\right)} \tag{7}$$

and

$$K_{A_2}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi\mu b_2}} e^{\frac{1}{2}\mu\left(\frac{a_2}{b_2}{x_2}^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}{\omega_2}^2\right)} \tag{8}$$

where μ stands for an unit pure quaternion so that $\mu^2 = -1$.

Here $e^{\mu\left(\frac{c_1 d_1}{2}\right){\omega_1}^2}$ and $e^{\mu\left(\frac{c_2 d_2}{2}\right){\omega_2}^2}$ are called chirp signals in signal processing. From hence we will deal with the case when $b_1 b_2 \neq 0$, because $L^{\mathbb{H}}_{A_1 A_2}\{f\}(\boldsymbol{\omega})$ is trivial for $b_1 b_2 = 0$.

As a special case, when $A_1 = A_2(a_i, b_i, c_i, d_i) = (0,1,-1,0)$ for $i = 1,2$, the LCT definition (6) reduces to the QFT definition (see [2, 3]. That is

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) \frac{1}{\sqrt{2\pi\mu}} e^{-\mu\omega_1 x_1} \frac{1}{\sqrt{2\pi\mu}} e^{-\mu\omega_2 x_2}\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \frac{1}{2\pi\mu} \\ &= \mathcal{F}_q\{f\}(\boldsymbol{\omega}) \frac{1}{2\pi\mu}, \end{aligned} \tag{9}$$

where the QFT of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is defined by

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}} d\boldsymbol{x}. \tag{10}$$

The following lemma demonstrates the general relationship between the QLCT and the QFT of a signal f.

Lemma 3.2 The QLCT of a signal f with $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ can be seen as the QFT of signal f in the form

$$\begin{aligned} &L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \\ &= \mathcal{F}_q\left\{f(\boldsymbol{x}) e^{\mu\frac{a_1}{2b_1}x_1^2} e^{\mu\frac{a_2}{2b_2}x_2^2}\right\}\left(\frac{\omega_1}{b_1}, \frac{\omega_2}{b_2}\right) \frac{1}{\sqrt{-2\pi\mu b_1}} \frac{1}{\sqrt{-2\pi\mu b_2}} e^{\mu\frac{d_1}{2b_1}\omega_1^2} e^{\mu\frac{d_2}{2b_2}\omega_2^2}. \end{aligned} \tag{11}$$

The inverse transform of the QLCT is given by

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) K_{A_1^{-1}}(x_1,\omega_1) K_{A_2^{-1}}(x_2,\omega_2)\, d\boldsymbol{\omega}.$$

Or, equivalently,

$$f(x) = \int_{\mathbb{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \frac{1}{\sqrt{-2\pi\mu b_1}} \frac{1}{\sqrt{-2\pi\mu b_2}} e^{-\frac{1}{2}\mu\left(\frac{d_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \times e^{-\frac{1}{2}\mu\left(\frac{d_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} d\boldsymbol{\omega}, \tag{13}$$

where
$A_1{}^{-1} = (d_1, -b_1, -c_1, a_1)$ and $A_2{}^{-1} = (d_2, -b_2, -c_2, a_2)$.

4. Application

In this section, we introduce an application of the QLCT to study of the generalized swept-frequency filters (compare to [6]). These filters are linear time-varying systems. The output of generalized swept-frequency filters is given by

$$\begin{aligned} y(\boldsymbol{x}) &= \left[f(\boldsymbol{x}) * g(\boldsymbol{x}) \left(e^{-\mu\frac{c_1}{2}x_1^2} e^{-\mu\frac{c_2}{2}x_2^2} \right) \right] e^{\mu\frac{c_1}{2}x_1^2} e^{\mu\frac{c_2}{2}x_2^2} \\ &= \int_{\mathbb{R}^2} \left[f(\boldsymbol{x}-\boldsymbol{\tau}) g(\boldsymbol{\tau}) e^{-\mu\frac{c_1}{2}\tau_1^2} e^{-\mu\frac{c_2}{2}\tau_2^2} \right] d\boldsymbol{\tau}\, e^{\mu\frac{c_1}{2}x_1^2} e^{\mu\frac{c_2}{2}x_2^2}, \end{aligned} \tag{14}$$

where "$*$ " is the traditional convolution operator. Here $g(\boldsymbol{x})$ is the impulse response of the shift-invariant filter. Choosing the matrix parameters $A_1 = (-c_1, 1, -1, 0)$ and $A_2 = (-c_2, 1, -1, 0)$ and then taking the LCT of both sides in the above equation we have

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[f(\boldsymbol{x}-\boldsymbol{\tau}) g(\boldsymbol{\tau}) e^{-\mu\frac{c_1}{2}\tau_1^2} e^{-\mu\frac{c_2}{2}\tau_2^2} \right] e^{\mu\frac{c_1}{2}x_1^2} e^{\mu\frac{c_2}{2}x_2^2} \\ &\quad \times \frac{1}{\sqrt{2\pi b_1\mu}} \frac{1}{\sqrt{2\pi b_2\mu}} e^{-\mu\frac{c_1}{2}x_1^2} e^{-\mu x_1\omega_1} e^{\mu\frac{c_2}{2}x_2^2} e^{-\mu x_2\omega_2} d\boldsymbol{x} d\boldsymbol{\tau} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[f(\boldsymbol{x}-\boldsymbol{\tau}) g(\boldsymbol{\tau}) e^{-\mu\frac{c_1}{2}\tau_1^2} e^{-\mu\frac{c_2}{2}\tau_2^2} \right] e^{\mu\frac{c_1}{2}x_1^2} e^{\mu\frac{c_2}{2}x_2^2} \\ &\quad \times \frac{1}{\sqrt{2\pi b_1\mu}} \frac{1}{\sqrt{2\pi b_2\mu}} e^{-\mu x_1\omega_1} e^{-\mu x_2\omega_2} d\boldsymbol{x} d\boldsymbol{\tau}. \end{aligned}$$

By making the change of variable $\boldsymbol{x} - \boldsymbol{\tau} = \boldsymbol{y}$ we immediately obtain

$$L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\boldsymbol{y}) g(\boldsymbol{\tau}) e^{-\mu\frac{c_1}{2}\tau_1^2} e^{-\mu\frac{c_2}{2}\tau_2^2}$$

$$\times \frac{1}{\sqrt{2\pi b_1\mu}}\frac{1}{\sqrt{2\pi b_2\mu}}e^{-\mu\omega_1\tau_1}e^{-\mu\omega_2\tau_2}e^{-\mu\omega_1 y_1}e^{-\mu\omega_2 y_2}d\boldsymbol{y}d\boldsymbol{\tau}$$

$$= \int_{\mathbb{R}^2} f(\boldsymbol{y})L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\, e^{-\mu y_1\omega_1}e^{-\mu y_2\omega_2}d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} \big(f_0(\boldsymbol{y}) + \boldsymbol{i}f_1(\boldsymbol{y}) + \boldsymbol{j}f_2(\boldsymbol{y}) + \boldsymbol{k}f_3(\boldsymbol{y})\big)\, L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})e^{-\mu y_1\omega_1}e^{-\mu y_2\omega_2}d\boldsymbol{y}$$

$$= \int_{\mathbb{R}^2} \big(f_0(\boldsymbol{y}) + \boldsymbol{i}f_1(\boldsymbol{y}) + \boldsymbol{j}f_2(\boldsymbol{y}) + \boldsymbol{k}f_3(\boldsymbol{y})\big)\, L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})e^{-\mu y_1\omega_1}e^{-\mu y_2\omega_2}d\boldsymbol{y}$$

$$= L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_0(\boldsymbol{y})\, e^{-\mu\omega_1 y_1}e^{-\mu\omega_2 y_2}d\boldsymbol{y}$$

$$+\boldsymbol{i}\, L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_1(\boldsymbol{y})\, e^{-\mu\omega_1 y_1}e^{-\mu\omega_2 y_2}d\boldsymbol{y}$$

$$+\boldsymbol{j}\, L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_2(\boldsymbol{y})\, e^{-\mu\omega_1 y_1}e^{-\mu\omega_2 y_2}d\boldsymbol{y}$$

$$+\boldsymbol{k}\, L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} f_3(\boldsymbol{y})\, e^{-\mu\omega_1 y_1}e^{-\mu\omega_2 y_2}d\boldsymbol{y}.$$

By the QLCT definition, we conclude therefore that

$$\begin{aligned} L^{\mathbb{H}}_{A_1 A_2}\{y\}(\boldsymbol{\omega}) = {} & L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \boldsymbol{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}) \\ & +\boldsymbol{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) + \boldsymbol{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega}). \end{aligned}$$

We call $L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})$ as the transfer function of the generalized swept-frequency filter in the QLCT domain. Equation mentioned above shows that the use of the QLCT generalizes the treatment of swept-frequency filters from the classical treatment of shift-invariant filters with the Fourier transform to the shift-invariant filters with the QLCT.

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